Algebras Defined by Monic Gröbner Bases over Rings *

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Abstract. Let $K\langle X\rangle = K\langle X_1,...,X_n\rangle$ be the free algebra of n generators over a field K, and let $R\langle X\rangle = R\langle X_1,...,X_n\rangle$ be the free algebra of n generators over an arbitrary commutative ring R. In this semi-expository paper, it is clarified that any monic Gröbner basis in $K\langle X\rangle$ may give rise to a monic Gröbner basis of the same type in $R\langle X\rangle$, and vice versa. This fact turns out that many important R-algebras have defining relations which form a monic Gröbner basis, and consequently, such R-algebras may be studied via a nice PBW structure theory as that developed for quotient algebras of $K\langle X\rangle$ in ([LWZ], [Li2, 3]).

0. Introduction

In the structure theory and the representation theory of associative algebras over a ground field K, it is well known that numerous popularly studied algebras have defining relations which form a Gröbner basis in the classical sense (e.g., [Mor], [Gr]), and such algebras can be studied in a computational way via their Gröbner defining relations (e.g., see [An], [CU], [GI-L], [Gr], [Li2, 3], [Uf1, 2]); also we know that algebras defined by the relations of the same type over a commutative ring R are equally important, for instance, the algebras over rings considered in [Yam], [Ber], [CE], and [LVO2]. So, naturally we expect that certain algebras over rings could be studied by means of Gröbner basis theory as in loc. cit., and thus we hope that the following statement would hold true:

- Let $K\langle X\rangle = K\langle X_1,...,X_n\rangle$ be the free algebra of n generators over a field K, and let $R\langle X\rangle = R\langle X_1,...,X_n\rangle$ be the free algebra of n generators over an arbitrary commutative ring R. By a Gröbner basis for an ideal in a free algebra we mean the one as defined in [Mor] and [Gr]. If, with respect to some monomial ordering $\prec$ on $K\langle X\rangle$, a subset $\mathcal{G} \subset K\langle X\rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$, then, taking a counterpart of $\mathcal{G}$ in $R\langle X\rangle$ (if

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it exists, again denoted by $\mathcal{G}$) and using the same monomial ordering $\prec$ on $R\langle X\rangle$, $\mathcal{G}$ is a Gröbner basis for the ideal $\langle\mathcal{G}\rangle$ in $R\langle X\rangle$.

To see at what level the above statement may hold true, it is necessary to see whether a version of the classical termination theorem ([Mor], [Gr]) works well for verifying Gröbner bases in $R\langle X\rangle$.

Let $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ be the free associative K-algebra of n generators over a field K, and let $\mathcal{B}$ be the standard K-basis of $K\langle X\rangle$ consisting of monomials (words in alphabet $X = \{X_1, ..., X_n\}$, including empty word which is identified with the multiplicative identity element 1 of $K\langle X\rangle$). Given a monomial ordering $\prec$ on $\mathcal{B}$ (i.e. a well-ordering $\prec$ on $\mathcal{B}$ satisfying: $u \prec v$ implies $wus \prec wvs$ for all $w, u, v, s \in \mathcal{B}$), and $f, g \in K\langle X\rangle - \{0\}$, if there are monomials $u, v \in \mathcal{B}$ such that

(1) $\mathbf{LM}(f)u = v\mathbf{LM}(g)$, and

(2) $\mathbf{LM}(f) \not| \, v$ and $\mathbf{LM}(g) \not| \, u$,

then the element

$$o(f, u;\ v, g) = \frac{1}{\mathbf{LC}(f)}(f \cdot u) - \frac{1}{\mathbf{LC}(g)}(v \cdot g)$$

is referred to as an overlap element of f and g, where, with respect to $\prec$, $\mathbf{LM}(\ )$ denotes the function taking the leading monomial and $\mathbf{LC}(\ )$ denotes the function taking the leading coefficient on elements of $K\langle X\rangle$ respectively. Over the ground field K, the termination theorem in the sense of ([Mor], [Gr]), which is known an algorithmic version of Bergman's diamond lemma [Ber1], states that

- if $\mathcal{G}$ is an LM-reduced subset of $K\langle X\rangle$ (i.e., $\mathbf{LM}(g_i) \not| \, \mathbf{LM}(g_j)$ for $g_i, g_j \in \mathcal{G}$ with $i \neq j$), then $\mathcal{G}$ is a Gröbner basis for the ideal $I = \langle\mathcal{G}\rangle$ if and only if for each pair $g_i, g_j \in \mathcal{G}$, including $g_i = g_j$, every overlap element $o(g_i, u;\ v, g_j)$ of g_i and g_j has the property $\overline{o(g_i, u;\ v, g_j)}^{\mathcal{G}} = 0$, that is, $o(g_i, u;\ v, g_j)$ is reduced to 0 by division by $\mathcal{G}$;

and it follows that there is a noncommutative analogue of the Buchberger algorithm for constructing a (possibly infinite) Gröbner basis starting with a given finite subset in $K\langle X\rangle$. Note that the algorithmic feasibility of the above criterion lies in the fact that

(a) for each pair (g_i, g_j) there are only finitely many associated overlap elements, and

(b) there is no trouble with taking the inverse of a nonzero coefficient when the division algorithm is performed, for, K is a field.

However, if the field K is replaced by a commutative ring R, and if $\mathcal{G} \subset R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ is taken such that the leading coefficient $\mathbf{LC}(g)$ of some $g \in \mathcal{G}$ is not invertible, then, even if R is an arithmetic ring (e.g. the ring $\mathbb{Z}$ of integers) as recently considered by [Gol], there seems no implementable termination theorem (as we mentioned above) for $\mathcal{G}$. Nevertheless, we have the following observations:

(1) If $\mathcal{G}$ is a Gröbner basis for an ideal I in the free algebra $K\langle X\rangle$ over a field K, then we may always assume that all elements of $\mathcal{G}$ are monic, i.e., $\mathbf{LC}(g) = 1$ for every $g \in \mathcal{G}$.

(2) If S is a subset consisting of monic elements in the free algebra $R\langle X\rangle$ over a ring R, i.e., $\mathbf{LC}(f) = 1$ for every $f \in S$, then, with respect to any monomial ordering $\prec$ on $R\langle X\rangle$, a

division algorithm by S can be implemented in $R\langle X\rangle$ exactly as in $K\langle X\rangle$.

Recalling the proof of the classical termination theorem ([Mor], [Gr]), the above observations provide us with sufficient reason to have an implementable termination theorem (as mentioned above) for verifying whether certain given monic elements of $R\langle X\rangle$ form a Gröbner basis in the classical sense, so that our foregoing expectation may come true. To present the details, we organize this paper as follows. In Section 1, after giving a quick introduction of the notion of a monic Gröbner basis in $R\langle X\rangle$, we examine carefully that a version of the termination theorem in the sense of ([Mor], [Gr]) holds true for verifying LM-reduced monic Gröbner bases in $R\langle X\rangle$, just for convincing ourselves and also for the reader's convenience from the viewpoint of "to see is to believe". This enables us to clarify that every monic Gröbner basis in the free algebra $K\langle X\rangle$ over a field K may give rise to a monic Gröbner basis of the same type in the free algebra $R\langle X\rangle$, and vice versa. In Section 2, by strengthening and generalizing a result of [Li2], we show how PBW R-bases and monic Gröbner bases of certain type can determine each other. In the final Section 3, we show that the working principle via PBW isomorphism developed in [LWZ] and [Li3] can be generalized to study quotient algebras of $R\langle X\rangle$, so that many global structural properties of R-algebras defined by monic Gröbner bases may be determined through their $\mathbb{N}$-leading homogeneous algebras and $\mathcal{B}_R$-leading homogeneous algebras.

Unless otherwise stated, rings considered in this paper are associative rings with multiplicative identity 1, ideals are meant two-sided ideals, and modules are unitary left modules. For a subset U of a ring S, we write $\langle U\rangle$ (or ${}_S\langle U\rangle_S$ if necessary) for the ideal generated by U. Moreover, we use $\mathbb{N}$, respectively $\mathbb{Z}$, to denote the set of nonnegative integers, respectively the set of integers.

1. Monic Gröbner Bases over K vs Monic Gröbner Bases over R

Let R be an arbitrary commutative ring, $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the free R-algebra of n generators, and $\mathcal{B}_R$ the standard R-basis of $R\langle X\rangle$ consisting of monomials (words in alphabet $X = \{X_1, ..., X_n\}$, including empty word which is identified with the multiplicative identity element 1 of $R\langle X\rangle$). Unless otherwise stated, monomials in $\mathcal{B}_R$ are denoted by lower case letters $u, v, w, s, t, \cdots$. In this section, after introducing the notion of a monic Gröbner basis in $R\langle X\rangle$ and examining carefully that a version of the termination theorem in the sense of ([Mor], [Gr]) holds true for verifying LM-reduced monic Gröbner bases in $R\langle X\rangle$, we clarify that every monic Gröbner basis in the free algebra $K\langle X_1, ..., X_n\rangle$ over a field K has a counterpart in the free algebra $R\langle X\rangle$, and vice versa.

First note that all monomial orderings used for free algebras over a field can be well defined on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$. In particular, by an $\mathbb{N}$-*graded monomial ordering* on $\mathcal{B}_R$, denoted $\prec_{gr}$, we mean a monomial ordering on $\mathcal{B}_R$ which is defined subject to a well-ordering

$\prec$ on $\mathcal{B}_R$, that is, for $u, v \in \mathcal{B}_R$, $u \prec_{gr} v$ if either $\deg u < \deg v$ or $\deg u = \deg v$ but $u \prec v$, where deg( ) denotes the degree function on elements of $R\langle X\rangle$ with respect to a fixed *weight* $\mathbb{N}$-*gradation* of $R\langle X\rangle$ (i.e. each X_i is assigned a positive degree n_i, $1 \le i \le n$). For instance, the usual $\mathbb{N}$-graded (reverse) lexicographic ordering is a popularly used $\mathbb{N}$-graded monomial ordering.

If $\prec$ is a monomial ordering on $\mathcal{B}_R$ and $f = \sum_{i=1}^{s} \lambda_i w_i \in R\langle X\rangle$, where $\lambda_i \in R - \{0\}$ and $w_i \in \mathcal{B}_R$, such that $w_1 \prec w_2 \prec \cdots \prec w_s$, then the *leading monomial* of f is defined as $\mathbf{LM}(f) = w_s$ and the *leading coefficient* of f is defined as $\mathbf{LC}(f) = \lambda_s$. For a subset $H \subset R\langle X\rangle$, we write $\mathbf{LM}(H) = \{\mathbf{LM}(f) \mid f \in H\}$ for the set of leading monomials of S. We say that a subset $G \subset R\langle X\rangle$ is *monic* if $\mathbf{LC}(g) = 1$ for all $g \in G$. Moreover, for $u, v \in \mathcal{B}_R$, as usual we say that v divides u, denoted $v|u$, if $u = wvs$ for some $w,\ s \in \mathcal{B}_R$.

With notation and all definitions as above, it is easy to see that a division algorithm by a monic subset G is valid in $R\langle X\rangle$ with respect to any fixed monomial ordering $\prec$ on $\mathcal{B}_R$. More precisely, let $f \in R\langle X\rangle$. Noticing $\mathbf{LC}(g) = 1$ for all $g \in G$, if $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in G$, then f can be written as $f = \mathbf{LC}(f)ugv + f_1$ with $u, v \in \mathcal{B}_R$, $f_1 \in R\langle X\rangle$ satisfying $\mathbf{LM}(f_1) \prec \mathbf{LM}(f)$; if $\mathbf{LM}(g) \not| \mathbf{LM}(f)$ for all $g \in G$, then $f = f_1 + \mathbf{LC}(f)\mathbf{LM}(f)$ with $f_1 = f - \mathbf{LC}(f)\mathbf{LM}(f)$ satisfying $\mathbf{LM}(f_1) \prec \mathbf{LM}(f)$. Next, consider the divisibility of $\mathbf{LM}(f_1)$ by $\mathbf{LM}(g)$ with $g \in G$, and so forth. Since $\prec$ is a well-ordering, after a finite number of successive division by elements in G in this way, we see that f can be written as

$$\begin{aligned} f \;=\; & \textstyle\sum_{i,j} \lambda_{ij} u_{ij} g_j v_{ij} + r_f, \text{ where } \lambda_{ij} \in R,\ u_{ij}, v_{ij} \in \mathcal{B}_R,\ g_j \in G, \\ & \text{and } r_f = \textstyle\sum_p \lambda_p w_p \text{ with } \lambda_p \in R,\ w_p \in \mathcal{B}_R, \\ & \text{satisfying } \mathbf{LM}(u_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0, \\ & \mathbf{LM}(r_f) \preceq \mathbf{LM}(f) \text{ and } \mathbf{LM}(g) \not| w_p \text{ for every } g \in G \text{ whenever } \lambda_p \neq 0. \end{aligned}$$

If, in the presentation of f above, $r_f = 0$, then we say that f *is reduced to* 0 *by division by* G, and we write $\overline{f}^G = 0$ for this property.

The validity of such a division algorithm by G leads to the following definition.

1.1. Definition Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$, and I an ideal of $R\langle X\rangle$. A *monic Gröbner basis* of I is a subset $\mathcal{G} \subset I$ satisfying:
(1) $\mathcal{G}$ is monic; and
(2) $f \in I$ and $f \neq 0$ implies $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in \mathcal{G}$.

By the division algorithm presented above, it is clear that a monic Gröbner basis of I is first of all a generating set of the ideal I, and moreover, a monic Gröbner basis of I can be characterized as follows.

1.2. Proposition Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$, and I an ideal of $R\langle X\rangle$. For a monic subset $\mathcal{G} \subset I$, the following statements are equivalent:
(i) $\mathcal{G}$ is a monic Gröbner basis of I;

(ii) Each nonzero $f \in I$ has a Gröbner representation:

$$\begin{array}{rcl} f & = & \sum_{i,j} \lambda_{ij} u_{ij} g_j v_{ij}, \text{ where } \lambda_{ij} \in R,\ u_{ij}, v_{ij} \in \mathcal{B}_R,\ g_j \in G, \\ & & \text{satisfying } \mathbf{LM}(u_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0, \end{array}$$

or equivalently, $\overline{f}^{\mathcal{G}} = 0$;
(iii) $\langle \mathbf{LM}(\mathcal{G})\rangle = \langle \mathbf{LM}(I)\rangle$. $\square$

Let $\prec$ be a monomial ordering on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, and let G be a monic subset of $R\langle X\rangle$. We call an element $f \in R\langle X\rangle$ a *normal element* (mod G) if $f = \sum_j \mu_j v_j$ with $\mu_j \in R$, $v_j \in \mathcal{B}_R$, and f has the property that $\mathbf{LM}(g) \not| \, v_j$ for every $g \in G$ and all $\mu_j \neq 0$. The set of normal monomials in $\mathcal{B}_R$ (mod G) is denoted by $N(G)$, i.e.,

$$N(G) = \{u \in \mathcal{B}_R \mid \mathbf{LM}(g) \not| \, u,\ g \in G\}.$$

Thus, an element $f \in R\langle X\rangle$ is normal (mod G) if and only if $f \in \sum_{u \in N(G)} Ru$.

1.3. Proposition Let $\mathcal{G}$ be a monic Gröbner basis of the ideal $I = \langle \mathcal{G}\rangle$ in $R\langle X\rangle$ with respect to some monomial ordering $\prec$ on $\mathcal{B}_R$. Then each nonzero $f \in R\langle X\rangle$ has a finite presentation

$$f = \sum_{i,j} \lambda_{ij} s_{ij} g_i w_{ij} + r_f, \quad \lambda_{ij} \in R,\ s_{ij}, w_{ij} \in \mathcal{B}_R,\ g_i \in \mathcal{G},$$

where $\mathbf{LM}(s_{ij} g_i w_{ij}) \preceq \mathbf{LM}(f)$ whenever $\lambda_{ij} \neq 0$, and either $r_f = 0$ or r_f is a unique normal element (mod $\mathcal{G}$). Hence, $f \in I$ if and only if $r_f = 0$, solving the "membership problem" for I.

Proof By the division algorithm by $\mathcal{G}$, f can be written as $f = \sum_{i,j} \lambda_{ij} s_{ij} g_i w_{ij} + r_f$ where either $r_f = 0$ or r_f is normal. Suppose after division by $\mathcal{G}$ we also have $f = \sum_{t,j} \lambda_{tj} s_{tj} g_t w_{tj} + r$, where r is normal (mod $\mathcal{G}$). Then $r - r_f \in I$ and hence there is some $g \in \mathcal{G}$ such that $\mathbf{LM}(g) | \mathbf{LM}(r - r_f)$. But by the definition of a normal element this is possible only if $r = r_f$. $\square$

The foregoing discussion enables us to obtain further characterization of a monic Gröbner basis $\mathcal{G}$, which, in turn, gives rise to the fundamental decomposition theorem of $R\langle X\rangle$ (viewed as an R-module) by the ideal $I = \langle \mathcal{G}\rangle$, and gives rise to a free R-basis for the R-algebra $R\langle X\rangle / I$.

1.4. Theorem Let $I = \langle \mathcal{G}\rangle$ be an ideal of $R\langle X\rangle$ generated by a monic subset $\mathcal{G}$. With notation as above, the following statements are equivalent.
(i) $\mathcal{G}$ is a monic Gröbner basis of I.
(ii) The R-module $R\langle X\rangle$ has the decomposition

$$R\langle X\rangle = I \oplus \sum_{u \in N(\mathcal{G})} Ru = \langle \mathbf{LM}(I)\rangle \oplus \sum_{u \in N(\mathcal{G})} Ru.$$

(iii) The canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $R\langle X\rangle / \langle \mathbf{LM}(I)\rangle$ and $R\langle X\rangle / I$ forms a free R-basis for $R\langle X\rangle / \langle \mathbf{LM}(I)\rangle$ and $R\langle X\rangle / I$ respectively.

$\square$

With notation and every definition introduced so far, we proceed now to show that a version of the termination theorem in the sense of ([Mor], [Gr]) holds true for verifying an LM-reduced monic Gröbner basis in $R\langle X\rangle$ (see the definition below).

Given a monomial ordering $\prec$ on $\mathcal{B}_R$, we say that a subset $G \subset R\langle X\rangle$ is *LM-reduced* if

$$\mathbf{LM}(g_i) \not| \, \mathbf{LM}(g_j) \text{ for all } g_i, g_j \in G \text{ with } g_i \neq g_j.$$

If a subset $G \subset R\langle X\rangle$ is both LM-reduced and monic, then we call G an *LM-reduced monic subset*. Thus we have the notion of an *LM-reduced monic Gröbner basis*.

Let I be an ideal of $R\langle X\rangle$. If $\mathcal{G}$ is a monic Gröbner basis of I and $g_1, g_2 \in \mathcal{G}$ such that $g_1 \neq g_2$ but $\mathbf{LM}(g_1)|\mathbf{LM}(g_2)$, then clearly g_2 can be removed from $\mathcal{G}$ and the remained subset $\mathcal{G} - \{g_2\}$ is again a monic Gröbner basis for I. Hence, in order to have a better criterion for monic Gröbner basis we need only to consider the subset which is both LM-reduced and monic.

Let $\prec$ be a monomial ordering on $\mathcal{B}_R$. For two monic elements $f,\ g \in R\langle X\rangle - \{0\}$, including $f = g$, if there are monomials $u, v \in \mathcal{B}_R$ such that
(1) $\mathbf{LM}(f)u = v\mathbf{LM}(g)$, and
(2) $\mathbf{LM}(f) \not| \, v$ and $\mathbf{LM}(g) \not| \, u$,
then the element

$$o(f, u;\ v, g) = f \cdot u - v \cdot g$$

is called an *overlap element* of f and g. From the definition we are clear about the fact that

$$\mathbf{LM}((o(f, u;\ v, g)) \prec \mathbf{LM}(fu) = \mathbf{LM}(vg),$$

and moreover, there are only finitely many overlap elements for each pair (f, g) of elements in $R\langle X\rangle$.

With the preparation made above, below we mention a termination theorem for checking LM-reduced monic Gröbner bases in $R\langle X\rangle$, and, also we present a careful step-by-step verification of its correctness for the reason that we are working on an arbitrary ring instead of a field after all, though the process is only a light modification of the argument given in [Gr].

1.5. Theorem (Termination theorem) Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$. If $\mathcal{G}$ is an LM-reduced monic subset of $R\langle X\rangle$, then $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $I = \langle\mathcal{G}\rangle$ if and only if for each pair $g_i, g_j \in \mathcal{G}$, including $g_i = g_j$, every overlap element $o(g_i, u;\ v, g_j)$ of $g_i,\ g_j$ has the property $\overline{o(g_i, u;\ v, g_j)}^{\mathcal{G}} = 0$, that is, by division by $\mathcal{G}$, every $o(g_i, u;\ v, g_j)$ is reduced to zero.

Proof Since $o(g_i, u;\ v, g_j) \in I$, the necessity follows from Proposition 1.2.

Under the assumption on overlap elements we prove the sufficiency by showing that if $f \in I$ then $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in \mathcal{G}$. Suppose the contrary that $\mathbf{LM}(g) \not| \mathbf{LM}(f)$ for any $g \in \mathcal{G}$. Then we proceed to derive a contradiction.

Since $I = \langle \mathcal{G} \rangle$, f may be presented as a finite sum

$$f = \sum_{i,j} \lambda_{ij} v_{ij} g_i w_{ij}, \text{ where } \lambda_{ij} \in R, \ v_{ij}, w_{ij} \in \mathcal{B}_R, \text{ and } g_i \in \mathcal{G}. \tag{1}$$

Let u be the largest monomial occurring on the right hand side of (1). Then noticing that each g_i is monic, u occurs as some $v_{ij}\mathbf{LM}(g_i)w_{ij}$. Since $\mathbf{LM}(g_i) \not| \mathbf{LM}(f)$ for the g_i occurring in (1), it follows that $\mathbf{LM}(f) \prec u$ and u must occur at least twice on the right hand side of (1) for a cancelation, that is, we may have

$$u = v_{ij}\mathbf{LM}(g_i)w_{ij} = v_{k\ell}\mathbf{LM}(g_k)w_{k\ell}. \tag{2}$$

Among all such presentations of f we can choose one such that

(3) u has the fewest occurrences on the right hand side of (1) and u is as small as possible.

To go further, let us simplify notation by writing $v = v_{ij}$, $g = g_i$, $w = w_{ij}$, $v' = v_{k\ell}$, $g' = g_k$, and $w' = w_{k\ell}$. Thus, the above (2) is turned into the form

$$u = v\mathbf{LM}(g)w = v'\mathbf{LM}(g')w'. \tag{4}$$

Moreover, as usual we use $l(s)$ to denote the length of a monomial $s \in \mathcal{B}_R$. Below we show, through a case by case study of the above (4), that the choice of f satisfying the above (3) is impossible.

Case A: $l(v) < l(v')$.

Case A.1: $l(w) < l(w')$.

This implies that $\mathbf{LM}(g)$ contains $\mathbf{LM}(g')$ as a subword, and hence, $\mathbf{LM}(g')|\mathbf{LM}(g)$, contradicting the hypothesis on $\mathcal{G}$.

Case A.2: $l(w) \ge l(w')$. Then we have to deal with two possibilities.

Case A.2.1: $l(v') \ge l(v\mathbf{LM}(g))$.

This implies that there is no overlap element of $\mathbf{LM}(g)$ and $\mathbf{LM}(g')$ in u. By the assumption on lengths, it follows that there is a segment w'' of v' such that $v' = v\mathbf{LM}(g)w''$ and $w = w''\mathbf{LM}(g')w'$, i.e.,

$$u = v\mathbf{LM}(g)w''\mathbf{LM}(g')w'.$$

Rewriting g as $g = \mathbf{LM}(g) + \sum \lambda_i u_i$, $g' = \mathbf{LM}(g') + \sum \mu_i u_i'$, then

$$\begin{aligned} vgw &= vgw''g'w' - vgw''\left(g' - \mathbf{LM}(g')\right)w' \\ &= v\mathbf{LM}(g)w''g'w' + \textstyle\sum \lambda_i v u_i w'' g' w' - \sum \mu_i v g w'' u_i' w' \\ &= v'g'w' + \textstyle\sum \lambda_i v u_i w'' g' w' - \sum \mu_i v g w'' u_i' w'. \end{aligned}$$

Thus, in writing vgw this way, the number of occurrences of u in the chosen presentation of f satisfying the above (3) can be reduced, a contradiction.

Case A.2.2: $l(v') < l(v\mathbf{LM}(g))$.
This implies that there is an overlap element of $\mathbf{LM}(g)$ and $\mathbf{LM}(g')$ in u, that is, there is a segment r of w and a segment s of v' such that $vs = v'$, $rw' = w$ and $\mathbf{LM}(g)r = s\mathbf{LM}(g')$. Hence,

$$(5) \qquad u = v\mathbf{LM}(g)rw' = vs\mathbf{LM}(g')w' \text{ and } o(g, r;\ s, g') = gr - sg'.$$

Furthermore, it follows from $gr = o(g, r;\ s, g') + sg'$ that

$$(6) \qquad vgw = vgrw' = v \cdot o(g, r;\ s, g') \cdot w' + v'g'w'.$$

By the assumption, $o(g, r;\ s, g')$ is reduced to 0 under the division by $\mathcal{G}$, namely,

$$(7) \qquad o(g, r;\ s, g') = \sum_{k,j} c_{kj}v_{kj}g_kw_{kj},\ v_{kj}, w_{kj} \in \mathcal{B}_R,\ g_k \in \mathcal{G},\ c_{kj} \in R,$$

satisfying

$$(8) \qquad \text{if } c_{kj} \neq 0 \text{ then } v_{kj}\mathbf{LM}(g_k)w_{kj} \prec \mathbf{LM}(g \cdot r) = \mathbf{LM}(s \cdot g').$$

Combining the above (5) – (8), once again we see that the number of occurrences of u in the chosen presentation of f satisfying the foregoing (3) can be reduced, a contradiction.
Case B: $l(v) = l(v')$.
This implies $\mathbf{LM}(g)|\mathbf{LM}(g')$ or $\mathbf{LM}(g')|\mathbf{LM}(g)$, which contradicts the assumption that $\mathcal{G}$ is LM-reduced.
Case C: $l(v) > l(v')$.
This is similar to Case 1. $\square$

Remark (i) Let $\mathcal{G}$ be an LM-reduced monic subset of $R\langle X\rangle$ and $I = \langle\mathcal{G}\rangle$. Since for each pair $g_i,\ g_j \in \mathcal{G}$, including $g_i = g_j$, there are only finitely many associated overlap elements, by Theorem 1.5 we can check effectively whether $\mathcal{G}$ is a Gröbner basis of I or not, when $\mathcal{G}$ is a finite subset.
(ii) Obviously, if $\mathcal{G} \subset R\langle X\rangle$ is an LM-reduced subset with the property that each $g \in \mathcal{G}$ has the leading coefficient $\mathbf{LC}(g)$ which is invertible in R, then Theorem 1.5 is also valid for $\mathcal{G}$.
(iii) It is obvious as well that Theorem 1.5 does not necessarily induce an analogue of the Buchberger algorithm as in the classical case.
(iv) It is not difficult to see that all discussion we presented so far is valid for getting monic Gröbner bases in a commutative polynomial ring $R[x_1, ..., x_n]$ over an arbitrary commutative ring R where overlap elements are replaced by S-polynomials.

Note that, throughout the proof of Theorem 1.5, nothing involves the invertibility of an element in the ring R; moreover, division algorithm by monic elements in a free algebra (over a field or over a commutative ring) never touches on the invertibility of a coefficient. Therefore, we

are now clear about the relation between monic Gröbner bases over a field and monic Gröbner bases over a commutative ring.

1.6. Proposition Let $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ be the free algebra of n generators over a field K, and let $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ be the free algebra of n generators over an arbitrary commutative ring R. With notation as before, fixing the same monomial ordering $\prec$ on both $K\langle X\rangle$ and $R\langle X\rangle$, the following statements hold.
(i) If a monic subset $\mathcal{G} \subset K\langle X\rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$, then, taking a counterpart of $\mathcal{G}$ in $R\langle X\rangle$ (if it exists), again denoted by $\mathcal{G}$, $\mathcal{G}$ is a monic Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$.
(ii) If a monic subset $\mathcal{G} \subset R\langle X\rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$, then, taking a counterpart of $\mathcal{G}$ in $K\langle X\rangle$ (if it exists), again denoted by $\mathcal{G}$, $\mathcal{G}$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$.

$\square$

From the literature we know that numerous well-known K-algebras over a field K, such as the n-th Weyl algebra over K, the enveloping algebra of a K-Lie algebra, a K-exterior algebra, a K-Clifford algebra, a down-up K-algebra, etc., all have defining relations that form an LM-reduced monic Gröbner basis in a free K-algebra. Hence, by Proposition 1.6, if the field K is replaced by a commutative ring R, then all of these R-algebras have defining relations that form an LM-reduced monic Gröbner basis in a free R-algebra. Below we give another example illustrating Theorem 1.5 and Proposition 1.6.

Example 1. Let R be a commutative ring. Consider in $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the subset $\mathcal{G} = \Omega \cup \mathcal{R}$ consisting of

$$\begin{aligned}&\Omega \subseteq \{g_i = X_i^p \mid 1 \le i \le n\} \text{ with } p \ge 2 \text{ a fixed integer,}\\ &\mathcal{R} = \{g_{ji} = X_jX_i - \lambda_{ji}X_iX_j \mid 1 \le i < j \le n\} \text{ with } \lambda_{ji} \in R,\\ &\qquad \text{that is, } \lambda_{ji} \text{ may be zero.}\end{aligned}$$

In the case that $R = K$ is a field, it was verified in ([Li4], Example 4) that, under the $\mathbb{N}$-graded lexicographic ordering $\prec_{gr}$ such that $X_1 \prec_{gr} X_2 \prec_{gr} \cdots \prec_{gr} X_n$, $\mathcal{G}$ forms an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G}\rangle$ in $K\langle X\rangle$. Hence, by Proposition 1.6, $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G}\rangle$ in $R\langle X\rangle$. Furthermore, the division by $\mathbf{LM}(\mathcal{G})$ yields

$$N(\mathcal{G}) = \left\{ X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} \;\middle|\; \alpha_i \in \mathbb{N} \text{ and } 0 \le \alpha_s \le p-1 \text{ if } X_s^p \in \Omega \right\}.$$

It follows from Theorem 1.4 that both the algebras $R\langle X\rangle/I$ and $R\langle X\rangle/\langle \mathbf{LM}(I)\rangle$ have the free R-basis

$$\overline{N(\mathcal{G})} = \left\{ \overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots \overline{X}_n^{\alpha_n} \;\middle|\; \alpha_i \in \mathbb{N} \text{ and } 0 \le \alpha_s \le p-1 \text{ if } X_s^p \in \Omega \right\},$$

where each $\overline{X}_i$ is the canonical image of X_i in $R\langle X\rangle/I$ and $R\langle X\rangle/\langle \mathbf{LM}(I)\rangle$ respectively.

Here let us point out that this example contains two families of special R-algebras, that is, in the case where $\Omega=\emptyset$, the R-algebra $R\langle X\rangle/I$ is similar to the coordinate ring of a quantum affine n-space over a field (such a quantum coordinate ring over a field is defined with all the $\lambda_{ji}\neq 0$); and in the case where $\Omega=\{g_i=X_i^2 \mid 1\le i\le n\}$, the algebra $R\langle X\rangle/I$ is similar to the quantum grassmannian (or quantum exterior) algebra over a field in the sense of [Man] (such a quantum grassmannian algebra over a field is defined with all the $\lambda_{ji}\neq 0$).

We finish this section by a useful corollary of Theorem 1.5.

1.7. Corollary Let R be a commutative ring and R' a subring of R with the same identity element 1. If we consider the free R-algebra $R\langle X\rangle = R\langle X_1,...,X_n\rangle$ and the free R'-algebra $R'\langle X\rangle = R'\langle X_1,...,X_n\rangle$, then the following two statements are equivalent for a subset $\mathcal{G}\subset R'\langle X\rangle$:
(i) $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $I=\langle\mathcal{G}\rangle$ in $R'\langle X\rangle$ with respect to some monomial ordering $\prec$ on the standard R'-basis $\mathcal{B}_{R'}$ of $R'\langle X\rangle$;
(ii) $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $J=\langle\mathcal{G}\rangle$ in $R\langle X\rangle$ with respect to the monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, where $\prec$ is the same monomial ordering used in (i).

Proof Let $\mathcal{G}\subset R'\langle X\rangle$ be an LM-reduced monic subset. Noticing that $B_R=B_{R'}$, each pair $g_i,g_j\in\mathcal{G}$ has the same set of overlap elements in both $R'\langle X\rangle$ and $R\langle X\rangle$. Moreover, noticing that performing the division of an overlap element $o(g_i,u;\ v,g_j)$ by $\mathcal{G}$ in both $R'\langle X\rangle$ and $R\langle X\rangle$ uses only coefficients from R'. It follows that $\overline{o(g_i,u;\ v,g_j)}^{\mathcal{G}}=0$ in $R'\langle X\rangle$ if and only if $\overline{o(g_i,u;\ v,g_j)}^{\mathcal{G}}=0$ in $R\langle X\rangle$. Hence the equivalence of (i) and (ii) is proved by the termination theorem for LM-reduced monic Gröbner bases in $R'\langle X\rangle$ and the termination theorem for LM-reduced monic Gröbner bases in $R\langle X\rangle$, respectively. $\square$

2. PBW R-bases vs Specific Monic Gröbner Bases

Let R be a commutative ring and $A=R[a_1,...,a_n]$ a finitely generated R-algebra with generators $a_1,...,a_n$. If the set $\mathscr{B}=\{a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n} \mid \alpha_j\in\mathbb{N}\}$ forms a free R-basis of A, that is, A is, as an R-module, free with the basis $\mathscr{B}$, then, in honor of the classical PBW (Poincaré-Birkhoff-Witt) theorem for enveloping algebras of Lie algebras over a ground field K, the set $\mathscr{B}$ is usually referred to as a *PBW R-basis* of A. Presenting A as a quotient algebra of the free R-algebra $R\langle X\rangle = R\langle X_1,...,X_n\rangle$, i.e., $A=R\langle X\rangle/I$ with I an ideal of $R\langle X\rangle$, the aim of this section is to show, under a mild condition, that A has a PBW R-basis is equivalent to that I has a specific monic Gröbner basis. This result enables us to obtain PBW R-bases by means of monic Gröbner bases on one hand; and on the other hand, since it is well known that in practice there are different ways to find a PBW basis of a given algebra provided it exists (e.g., [Ros], [Yam], [Rin], [Ber]), this result also enables us to obtain monic Gröbner bases via already known PBW R-bases.

Throughout this section, we let R be a commutative ring, $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the free R-algebra of n generators, and $\mathcal{B}_R$ the standard R-basis of $R\langle X\rangle$. All notations and notions concerning monic Gröbner bases in $R\langle X\rangle$ are maintained as before.

Let I be an ideal of $R\langle X\rangle$ such that the R-algebra $A = R\langle X\rangle/I$ has the PBW R-basis $\mathscr{B} = \left\{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots\overline{X}_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\right\}$, where each $\overline{X}_i$ is the canonical image of X_i in A. Then I contains necessarily a subset G consisting of $\frac{n(n-1)}{2}$ elements of the form:

$$g_{ji} = X_jX_i - \sum_{\alpha}\lambda_\alpha w_\alpha,\ \text{where } 1 \le i < j \le n,\ \lambda_\alpha \in R,\ w_\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}.$$

In light of Theorem 1.4 and the observation made above, below we give the main result of this section which, indeed, strengthens and generalizes ([Li2], CH.III, Theorem 1.5).

2.1. Theorem Let I be an ideal of $R\langle X\rangle$, $A = R\langle X\rangle/I$. Suppose that I contains a monic subset of $\frac{n(n-1)}{2}$ elements $G = \{g_{ji} \mid 1 \le i < j \le n\}$ such that, with respect to some monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, $\mathbf{LM}(g_{ji}) = X_jX_i$ for $1 \le i < j \le n$. The following two statements are equivalent.
(i) The R-algebra A has the PBW R-basis $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots\overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ where each $\overline{X}_i$ is the canonical image of X_i in A.
(ii) Any monic subset $\mathcal{G}$ of I containing G is a monic Gröbner basis for I with respect to $\prec$.

Proof (i) $\Rightarrow$ (ii) Let $\mathcal{G}$ be a monic subset of I containing G, and let

$$N(\mathcal{G}) = \{u \in \mathcal{B}_R \mid \mathbf{LM}(g) \nmid u,\ g \in \mathcal{G}\}$$

be the set of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$). If $f \in I$ and $f \ne 0$, then, after implementing the division of f by $\mathcal{G}$ (with respect to the given monomial ordering $\prec$) we have

$$\begin{array}{rcl} f &=& \sum_{i,j}\lambda_{ij}u_{ij}g_iv_{ij} + r_f,\ \text{where } \lambda_{ij} \in R,\ u_{ij}, v_{ij} \in \mathcal{B}_R,\ g_i \in \mathcal{G}, \\ && \text{satisfying } \mathbf{LM}(w_{ij}g_iv_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \ne 0, \\ && \text{and } r_f = \sum_p \lambda_p w_p \text{ with } \lambda_p \in R \text{ and } w_p \in N(\mathcal{G}). \end{array}$$

Note that $g_{ji} \in G \subseteq \mathcal{G}$ and $\mathbf{LM}(g_{ji}) = X_jX_i$ by the assumption. It follows that $N(\mathcal{G}) \subseteq \{X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$. Thus, since $\mathscr{B}$ is a free R-basis of A, $r_f = \sum_p \lambda_p w_p = f - \sum_{i,j} u_{ij}g_iv_{ij} \in I$ implies $\lambda_p = 0$ for all p. Consequently $r_f = 0$. This shows that every nonzero element of I has a Gröbner representation by the elements of $\mathcal{G}$. Hence $\mathcal{G}$ is a monic Gröbner basis for I by Proposition 1.2.
(ii) $\Rightarrow$ (i) By (ii), the subset G itself is a monic Gröbner basis of I with respect to $\prec$. Let $N(G)$ be the set of normal monomials in $\mathcal{B}_R$ (mod G). Noticing that $\mathbf{LM}(g_{ji}) = X_jX_i$ for every $g_{ji} \in G$, it follows that $N(G) = \{X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$, and thereby the algebra A has the desired PBW R-basis $\mathscr{B}$ by Theorem 1.4. $\square$

We illustrate Theorem 2.1 by several examples. The first four examples given below serve to obtain monic Gröbner bases by means of already known PBW R-bases which are obtained in the literature without using the theory of Gröbner basis.

Example 1. (This is a special case of Example 3 given later.) Let $\mathsf{g} = R[V]$ be the R-Lie algebra defined by the free R-module $V = \oplus_{i=1}^{n} Rx_i$ and the bracket product $[x_j, x_i] = \sum_{\ell=1}^{n} \lambda_{ji}^{\ell} x_\ell$, $1 \le i < j \le n$, $\lambda_{ji}^{\ell} \in R$. By the classical PBW theorem, the universal enveloping algebra $U(\mathsf{g})$ of g has the PBW R-basis $\mathscr{B} = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$. If, with respect to the natural $\mathbb{N}$-gradation of $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$, we use an $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$ such that $X_1 \prec_{gr} X_2 \prec_{gr} \cdots \prec_{gr} X_n$ (i.e., $\deg X_i = 1$, $1 \le i \le n$), then the set of defining relations

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - X_i X_j - \sum_{\ell=1}^{n} \lambda_{ji}^{\ell} X_\ell \;\middle|\; 1 \le i < j \le n \right\}$$

of $U(\mathsf{g})$ satisfies $\mathbf{LM}(g_{ji}) = X_j X_i$ for $1 \le i < j \le n$. Hence, by Theorem 2.1, $\mathcal{G}$ is a monic Gröbner basis for the ideal $I = \langle \mathcal{G}\rangle$ in $R\langle X\rangle$.

Example 2. Let $U_q^+(A_N)$ be the (+)-part of the Drinfeld-Jimbo quantum group of type A_N over a commutative ring R, where $q \in R$ is invertible and $q^8 \ne 1$. This example shows that the defining relations (Jimbo relations) of $U_q^+(A_N)$ over R form a monic Gröbner basis in a free R-algebra. By Proposition 1.6, we reach this property over a field K.

In [Ros] and [Yam] it was proved that, over a field K, $U_q^+(A_N)$ has a PBW K-basis with respect to the defining relations (Jimbo relations) of $U_q^+(A_N)$; later in [BM] such a PBW basis was recaptured by showing that the Jimbo relations form a Gröbner basis ([BM] Theorem 4.1), where the proof was sketched to check that all compositions (overlaps) of Jimbo relations reduce to zero on the base argument of [Yam]. Recently, a very detailed elementary verification of the fact that all compositions (overlaps) of Jimbo relations reduce to zero and hence the Jimbo relations form a Gröbner basis (namely Theorem 4.1 of [BM]) was carried out by [CSS]. Now, by using Theorem 2.1 we will see that it is indeed very easy to conclude: the Jimbo relations form a Gröbner basis.

Recall that the Jimbo relations (as described in [Yam]) are given by

$$\begin{array}{ll}
x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}, & ((i,j),(m,n)) \in C_1 \cup C_3, \\
x_{mn}x_{ij} - x_{ij}x_{mn}, & ((i,j),(m,n)) \in C_2 \cup C_6, \\
x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}, & ((i,j),(m,n)) \in C_4, \\
x_{mn}x_{ij} - q^2 x_{ij}x_{mn} + qx_{in}, & ((i,j),(m,n)) \in C_1 \cup C_3,
\end{array}$$

where with $\Lambda_N = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i < j \le N+1\}$,

$$\begin{array}{ll}
C_1 = \{((i,j),(m,n)) \mid i = m < j < n\}, & C_2 = \{((i,j),(m,n)) \mid i < m < n < j\}, \\
C_3 = \{((i,j),(m,n)) \mid i < m < j = n\}, & C_4 = \{((i,j),(m,n)) \mid i < m < j < n\}, \\
C_5 = \{((i,j),(m,n)) \mid i < j = m < n\}, & C_6 = \{((i,j),(m,n)) \mid i < j < m < n\}.
\end{array}$$

By [Yam], for $q^8 \neq 1$, $U_q^+(A_N)$ has the PBW basis consisting of elements

$$x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} \text{ with } (i_1,j_1)\le (i_2,j_2)\le \cdots \le (i_k,j_k),\ k\ge 0,$$

where $(i_\ell, j_\ell)\in \Lambda_N$ and $<$ is the lexicographic ordering on Λ_N. If we use the $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ (on the standard K-basis $\mathcal{B}$ of the corresponding free algebra) subject to

$$x_{ij}\prec_{gr} x_{mn} \Longleftrightarrow (i,j)<(m,n),$$

then it is clear that for each pair $((i,j),(m,n))\in C_i$ with $(i,j)<(m,n)$, the leading monomial of the corresponding relation is of the form $x_{mn}x_{ij}$ as required by Theorem 2.1.

Example 3. With $R\langle X\rangle = R\langle X_1,...,X_n\rangle$, where R is an arbitrary commutative ring, recall from [Ber] that a q-algebra $A = R\langle X\rangle/\langle \mathcal{G}\rangle$ over R is defined by the set $\mathcal{G}$ of quadric relations

$$\begin{array}{rcl} g_{ji} &=& X_jX_i - q_{ji}X_iX_j - \{X_j,X_i\},\ 1\le i<j\le n,\ \text{where } q_{ji}\in R-\{0\},\\ && \text{and } \{X_j,X_i\} = \sum \alpha_{ji}^{k\ell}X_kX_\ell + \sum \alpha_hX_h + c_{ji},\ \alpha_{ji}^{k\ell},\alpha_h,c_{ji}\in R,\\ && \text{satisfying if } \alpha_{ji}^{kl}\neq 0,\ \text{then } i<k\le \ell<j,\ \text{and } k-i=j-\ell. \end{array}$$

Define two R-submodules of the free R-module $R\langle X\rangle$:

$$\begin{array}{rcl} \mathcal{E}_1 &=& R\text{-Span}\left\{ g_{ji}\ \middle|\ 1\le i<j\le n\right\},\\ \\ \mathcal{E}_2 &=& R\text{-Span}\left\{ X_ig_{ji},\ g_{ji}X_i,\ X_jg_{ji},\ g_{ji}X_j\ \middle|\ 1\le i<j\le n\right\}. \end{array}$$

If, for $1\le i<j<k\le n$, every Jacobi sum

$$\begin{array}{rcl} J(X_k,X_j,X_i) &=& \{X_k,X_j\}X_i - \lambda_{ki}\lambda_{ji}X_i\{X_k,X_j\}-\\ && -\lambda_{ji}\{X_k,X_i\}X_j + \lambda_{kj}X_j\{X_k,X_i\}+\\ && +\lambda_{kj}\lambda_{ki}\{X_j,X_i\}X_k - X_k\{X_j,X_i\} \end{array}$$

is contained in $\mathcal{E}_1+\mathcal{E}_2$, then A is called a q-*enveloping* algebra. Clearly, enveloping algebras of R-Lie algebras are special q-enveloping algebras with $q=1$. In [Ber], a q-PBW theorem for q-enveloping algebras over a commutative ring was obtained along the line similar to the classical argument on enveloping algebras of Lie algebras as given in [Jac], that is, if A is a q-enveloping R-algebra then A has the PBW R-basis $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots \overline{X}_n^{\alpha_n}\mid \alpha_j\in\mathbb{N}\}$.

Now, if we use the $\mathbb{N}$-graded monomial ordering $X_1\prec_{gr} X_2\prec_{gr}\cdots\prec_{gr} X_n$ on $\mathcal{B}_R$ with respect to the natural $\mathbb{N}$-gradation of $R\langle X\rangle$ (i.e., $\deg X_i = 1,\ 1\le i\le n$), then $\mathcal{G}$ satisfies $\mathbf{LM}(g_{ji}) = X_jX_i$ for all $1\le i<j\le n$. Hence, by Theorem 2.1, the set $\mathcal{G}$ of the defining relations of a q-enveloping R-algebra is a monic Gröbner basis for the ideal $I=\langle\mathcal{G}\rangle$ in $R\langle X\rangle$. In particular, all quantum algebras over $R=\mathbb{C}[[h]]$ which are q-enveloping algebras appeared in [Ber] are defined by monic Gröbner bases.

Remark It is necessary to point out that if $R=K$ is a field, then the fact that the set of defining relations $\mathcal{G}$ of a q-enveloping K-algebra A forms a Gröbner basis of the ideal $I=\langle\mathcal{G}\rangle$

was proved in ([Li2], CH.III) directly by using the termination theorem through the division algorithm. Here our last example provides the general result for all q-enveloping algebras over an arbitrary commutative ring.

Example 4. This example generalizes the previous three examples but uses an ad hoc monomial ordering. As an application we show that, over a commutative ring R, the PBW generators of the quantum algebra $U_q^+(A_N)$ derived in [Rin] provides another Gröbner defining set for $U_q^+(A_N)$.

With $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$, consider the R-algebra $A = R\langle X\rangle/\langle \mathcal{G}\rangle$ defined by the subset $\mathcal{G}$ consisting of $\frac{n(n-1)}{2}$ elements

$$\begin{aligned} g_{ji} &= X_jX_i - q_{ji}X_iX_j - \textstyle\sum_\alpha \lambda_\alpha X_{i_1}^{\alpha_1}X_{i_2}^{\alpha_2}\cdots X_{i_s}^{\alpha_s} + \lambda_{ji},\ 1\le i<j\le n,\\ &\ \text{where } q_{ji},\lambda_\alpha,\lambda_{ji}\in R,\ \alpha_k\in\mathbb{N},\ i<i_1\le i_2\le\cdots\le i_s<j.\end{aligned}$$

It is well-known that numerous iterated skew polynomial algebras over R are defined subject to such relations, and consequently they have the PBW R-basis $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots\overline{X}_n^{\alpha_n} \mid \alpha_j\in\mathbb{N}\}$. Under the assumption that A has the PBW R-basis as described we aim to show that $\mathcal{G}$ is a monic Gröbner basis of $\langle\mathcal{G}\rangle$. In view of Theorem 2.1, it is sufficient to introduce a monomial ordering on $\mathcal{B}_R$ so that $\mathbf{LM}(g_{ji}) = X_jX_i$ for all $1\le i<j\le n$. To this end, let $R[t] = R[t_1, ..., t_n]$ be the commutative polynomial R-algebra of n variables. Consider the canonical algebra epimorphism π: $R\langle X\rangle \to R[t]$ with $\pi(X_i) = t_i$. If we fix the lexicographic ordering $X_1 <_{lex} X_2 <_{lex} \cdots <_{lex} X_n$ on $\mathcal{B}_R$ of $R\langle X\rangle$ (note that $<_{lex}$ is not a monomial ordering on $\mathcal{B}_R$) and fix an arbitrarily chosen monomial ordering $\prec$ on the standard R-basis $\mathbb{B}_R = \{t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_n^{\alpha_n} \mid \alpha_j\in\mathbb{N}\}$ of $R[t]$, respectively, then, as in [EPS], a monomial ordering $\prec_{et}$ on $\mathcal{B}_R$, which is called the *lexicographic extension* of the given monomial ordering $\prec$ on $\mathbb{B}_R$, may be obtained as follows: for $u, v\in\mathcal{B}_R$,

$$u \prec_{et} v \text{ if } \begin{cases} \pi(u)\prec\pi(v),\\ \text{or}\\ \pi(u)=\pi(v) \text{ and } u <_{lex} v \text{ in } \mathcal{B}_R.\end{cases}$$

In particular, with respect to the monomial ordering $\prec_{et}$ obtained by using the lexicographic ordering $t_n \prec_{lex} t_{n-1} \prec_{lex} \cdots \prec_{lex} t_1$ on $\mathbb{B}_R$, we see that $\mathbf{LM}(g_{ji}) = X_jX_i$ for all $1\le i<j\le n$, as required by Theorem 2.1.

In [Rin] it was proved that $U_q^+(A_N)$ has $m = \frac{N(N+1)}{2}$ generators $x_1, ..., x_m$ satisfying the relations:

$$\begin{aligned} x_jx_i &= q^{v_{ji}}x_ix_j - r_{ji},\ 1\le i<j\le m,\ \text{where } v_{ji} = (wt(x_i), wt(x_j)),\ \text{and}\\ &\ r_{ji} \text{ is a linear combination of monomials of the form } x_{i+1}^{\alpha_{i+1}}x_{i+2}^{\alpha_{i+2}}\cdots x_{j-1}^{\alpha_{j-1}},\end{aligned}$$

and that $U_q^+(A_N)$ is an iterated skew polynomial algebra generated by $x_1, ..., x_m$ subject to the above relations. Thus $U_q^+(A_N)$ has the PBW basis $\{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} \mid \alpha_j\in\mathbb{N}\}$, and consequently $\mathcal{G} = \{g_{ji} = x_jx_i - q^{v_{ij}}x_ix_jr_{ji} \mid 1\le i<j\le m\}$ forms a monic Gröbner defining set of $U_q^+(A_N)$ with respect to the monomial ordering $\prec_{et}$ as described before.

Remark If, in the defining relations given in the last example, the condition $i < i_1 \le i_2 \le \cdots \le i_s < j$ is replaced by $1 \le i_1 \le i_2 \le \cdots \le i_s \le i-1$, then a similar result holds.

The next three examples provide monic Gröbner bases which are not necessarily the type as described in previous Examples 3 – 4, but they all give rise to PBW R-bases.

Example 5. Let R be a commutative ring, and let I be the ideal of the free R-algebra $R\langle X\rangle = R\langle X_1, X_2\rangle$ generated by the single element

$$g_{21} = X_2X_1 - qX_1X_2 - \alpha X_2 - f(X_1),$$

where $q, \alpha \in R$, and $f(X_1)$ is a polynomial in the variable X_1. Assigning to X_1 the degree 1, then in either of the following two cases:
(a) $\deg f(X_1) \le 2$, and X_2 is assigned the degree 1;
(b) $\deg f(X_1) = n \ge 3$, and X_2 is assigned the degree n,
$\mathcal{G} = \{g_{21}\}$ forms an LM-reduced monic Gröbner basis for I. For, in both cases we may use the $\mathbb{N}$-graded lexicographic ordering $X_1 \prec_{gr} X_2$ with respect to the natural $\mathbb{N}$-gradation of $K\langle X\rangle$, respectively the weight $\mathbb{N}$-gradation of $R\langle X\rangle$ with weight $\{1, n\}$, such that $\mathbf{LM}(g_{21}) = X_2X_1$, and then we see that the only overlap element of $\mathcal{G}$ is $o(g_{21}, 1;\ 1, g_{21}) = 0$. Thus, by Theorem 2.1 in both cases the algebra $A = R\langle X\rangle/I$ has the PBW R-basis $\mathscr{B} = \{\overline{X_1}^{\alpha}\overline{X_2}^{\beta} \mid \alpha, \beta \in \mathbb{N}\}$.

Example 6. Let R be a commutative ring, and let $R\langle X\rangle = R\langle X_1, X_2, X_3\rangle$ be the free R-algebra generated by $X = \{X_1, X_2, X_3\}$. This example provides a family of algebras similar to the enveloping algebra $U(\mathsf{sl}(2, R))$ of the R-Lie algebra $\mathsf{sl}(2, R)$, that is, we consider the algebra $A = K\langle X\rangle/\langle\mathcal{G}\rangle$ with $\mathcal{G}$ consisting of

$$\begin{aligned} g_{31} &= X_3X_1 - \lambda X_1X_3 + \gamma X_3, \\ g_{12} &= X_1X_2 - \lambda X_2X_1 + \gamma X_2, \\ g_{32} &= X_3X_2 - \omega X_2X_3 + f(X_1), \end{aligned}$$

where $\lambda, \gamma, \omega \in R$, and $f(X_1)$ is a polynomial in the variable X_1. It is clear that $A = U(\mathsf{sl}(2, R))$ in case $\lambda = \omega = 1$, $\gamma = 2$ and $f(X_1) = -X_1$.

Suppose $f(X_1)$ has degree $n \ge 1$. Then we can always equip $R\langle X\rangle$ with a weight $\mathbb{N}$-gradation by assigning to X_1, X_2 and X_3 the positive degree n_1, n_2, n_3 respectively (for instance, $(1, 1, 1)$ if $\deg f(X_1) = n \le 2$; $(1, n, n)$ if $\deg f(X_1) = n > 2$), such that $\mathbf{LM}(\mathcal{G}) = \{X_3X_1,\ X_1X_2,\ X_3X_2\}$ with respect to the $\mathbb{N}$-graded monomial ordering $X_2 \prec_{gr} X_1 \prec_{gr} X_3$ on $\mathcal{B}_R$. In the case that $R = K$ is a field, it was verified in ([Li4], Example 7) that $\mathcal{G}$ is a Gröbner basis for the ideal $\langle\mathcal{G}\rangle$ in $K\langle X\rangle$ with respect to the same $\prec_{gr}$. Hence, by Proposition 1.6, $\mathcal{G}$ is a Gröbner basis for the ideal $\langle\mathcal{G}\rangle$ in $R\langle X\rangle$. It follows from Theorem 2.1 that the algebra $A = R\langle X\rangle/\langle\mathcal{G}\rangle$ has the PBW R-basis $\mathscr{B} = \{\overline{X_2}^{\alpha_2}\overline{X_1}^{\alpha_1}\overline{X_3}^{\alpha_3} \mid \alpha_j \in \mathbb{N}\}$.

Let us point out that in the case that $f(X_1)$ has degree ≤ 2, i.e., $f(X_1)$ is of the form

$$f(X_1) = aX_1^2 + bX_1 + c \text{ with } a, b, c \in R,$$

if $\deg X_1 = \deg X_2 = \deg X_3 = 1$ is used, the algebra A provides R-versions of some popularly studied algebras over a field K, for instance,
(a) let $\zeta \in R$ be invertible, and put $\lambda = \zeta^4$, $\omega = \zeta^2$, $\gamma = -(1+\zeta^2)$, $a = 0 = c$, and $b = -\zeta$, then A is just the R-version of the Woronowicz's deformation of $U(\mathsf{sl}(2,K))$ introduced in the noncommutative differential calculus;
(b) if $\lambda\gamma wb \neq 0$ and $c = 0$, then A is just the R-version of Le Bruyn's conformal $\mathsf{sl}(2,K)$ enveloping algebra [LB] which provides a special family of Witten's deformation of $U(\mathsf{sl}(2,K))$ in quantum group theory.

Example 7. Let $\mathcal{G}$ be the subset of the free R-algebra $R\langle X\rangle = K\langle X_1, X_2, X_3\rangle$ consisting of

$$\begin{aligned} g_{21} &= X_2X_1 - X_1X_2, \\ g_{31} &= X_3X_1 - \lambda X_1X_3 - \mu X_2X_3 - \gamma X_2,\ \lambda, \mu, \gamma \in R, \\ g_{32} &= X_3X_2 - X_2X_3. \end{aligned}$$

Then, under the $\mathbb{N}$-graded lexicographic ordering $X_1 \prec_{gr} X_2 \prec_{gr} X_3$ with respect to the natural $\mathbb{N}$-gradation of $R\langle X\rangle$, $\mathbf{LM}(g_{ji}) = X_jX_i$, $1 \le i < j \le 3$, and the only nontrivial overlap element of $\mathcal{G}$ is $S_{321} = o(g_{32}, X_1;\ X_3, g_{21}) = -X_2X_3X_1 + X_3X_1X_2$. One checks easily that $\overline{S_{321}}^{\mathcal{G}} = 0$. By Theorem 2.1, $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $\langle\mathcal{G}\rangle$. Hence, by Theorem 2.1 the algebra $A = R\langle X\rangle/\langle\mathcal{G}\rangle$ has the PBW R-basis $\mathscr{B} = \{\overline{X_1}^{\alpha_1}\overline{X_2}^{\alpha_2}\overline{X_3}^{\alpha_3} \mid \alpha_j \in \mathbb{N}\}$.

3. PBW Isomorphisms and Applications

In this section we show that the working principle via PBW isomorphism developed in [LWZ] and [Li3] can be generalized to study algebras defined by monic Gröbner bases over a commutative ring R. All notions and notations used in previous sections are maintained.

Let R be an arbitrary commutative ring, $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the free R-algebra of n generators, and $\mathcal{B}_R$ the standard free R-basis of $R\langle X\rangle$. Consider a weight $\mathbb{N}$-gradation of $R\langle X\rangle$ subject to $\deg(X_i) = n_i > 0$, $1 \le i \le n$, that is, $R\langle X\rangle = \oplus_{p\in\mathbb{N}} R\langle X\rangle_p$ with $R\langle X\rangle_p = R$-span$\{w \in \mathcal{B} \mid \deg(w) = p\}$. For an element $f \in R\langle X\rangle$, say $f = F_0 + F_1 + \cdots + F_p$ with $F_i \in R\langle X\rangle_i$ and $F_p \neq 0$, let $\mathbf{LH}_{\mathbb{N}}(F)$ denote the $\mathbb{N}$-*leading homogeneous element* of f, i.e., $\mathbf{LH}_{\mathbb{N}}(f) = F_p$. Then every ideal I of $R\langle X\rangle$ is associated to an $\mathbb{N}$-graded ideal $\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle$ generated by the set of $\mathbb{N}$-leading homogeneous elements $\mathbf{LH}_{\mathbb{N}}(I) = \{\mathbf{LH}_{\mathbb{N}}(f) \mid f \in I\}$. Adopting the notion and notation as in [Li3], we call the $\mathbb{N}$-graded algebra $A^{\mathbb{N}}_{\mathrm{LH}} = R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle$ the $\mathbb{N}$-*leading homogeneous algebra* of the algebra $A = R\langle X\rangle/I$. On the other hand, noticing that $R\langle X\rangle$ is also a B_R-graded algebra by the multiplicative monoid B_R, i.e., $R\langle X\rangle = \oplus_{w\in\mathcal{B}_R} R\langle X\rangle_w$ with $R\langle X\rangle_w = Rw$, if $\prec$ is a monomial ordering on $\mathcal{B}_R$ and if $f = \sum_{i=1}^n \lambda_i w_i \in R\langle X\rangle$ with $w_1 \prec w_2 \prec \cdots \prec w_n$, then the term $\lambda_n w_n$ is called the $\mathcal{B}_R$-*leading homogeneous element* of f and is denoted by $\mathbf{LH}_{\mathcal{B}_R}(f)$. Thus each ideal I of $R\langle X\rangle$ is associated to a $\mathcal{B}_R$-graded ideal $\langle \mathbf{LH}_{\mathcal{B}_R}(I)\rangle$ generated by the set of $\mathcal{B}_R$-leading homogeneous elements $\mathbf{LH}_{\mathcal{B}_R}(I) = \{\mathbf{LH}_{\mathcal{B}_R}(f) \mid f \in I\}$, and similarly, the $\mathcal{B}_R$-graded

algebra $A^{\mathcal{B}_R}_{\rm LH} = R\langle X\rangle/\langle \mathbf{LH}_{\mathcal{B}_R}(I)\rangle$ is referred to as the $\mathcal{B}_R$-*leading homogeneous algebra* of the algebra $A = R\langle X\rangle/I$. Furthermore, consider the $\mathbb{N}$-grading filtration $F^{\mathbb{N}}R\langle X\rangle$ of $R\langle X\rangle$ defined by

$$F^{\mathbb{N}}_pR\langle X\rangle = \oplus_{i\le p}R\langle X\rangle_i,\quad p, i\in\mathbb{N},$$

and the $\mathcal{B}_R$-grading filtration $F^{\mathcal{B}_R}R\langle X\rangle$ of $R\langle X\rangle$ defined by

$$F^{\mathcal{B}_R}_wR\langle X\rangle = \oplus_{u\preceq w}R\langle X\rangle_u,\quad w, u\in\mathcal{B}_R.$$

If I is an ideal of $R\langle X\rangle$, then the algebra $A = R\langle X\rangle/I$ has the $\mathbb{N}$-filtration $F^{\mathbb{N}}A$ induced by $F^{\mathbb{N}}R\langle X\rangle$, i.e.,

$$F^{\mathbb{N}}_pA = (F^{\mathbb{N}}_pR\langle X\rangle + I)/I,\quad p\in\mathbb{N},$$

respectively the $\mathcal{B}_R$-filtration $F^{\mathcal{B}_R}A$ induced by $F^{\mathcal{B}_R}R\langle X\rangle$, i.e.,

$$F^{\mathcal{B}_R}_wA = (F^{\mathcal{B}_R}_wR\langle X\rangle + I)/I,\quad w\in\mathcal{B}_R.$$

Note that if each X_i has degree 1, $1\le i\le n$, then the filtration $F^{\mathbb{N}}A$ is just the commonly used *natural* $\mathbb{N}$-*filtration*. Let $G^{\mathbb{N}}(A) = \oplus_{p\in\mathbb{N}}G^{\mathbb{N}}(A)_p$ with $G^{\mathbb{N}}(A)_p = F^{\mathbb{N}}_pA/F^{\mathbb{N}}_{p-1}A$ be the associated $\mathbb{N}$-graded algebra of A determined by $F^{\mathbb{N}}A$, respectively $G^{\mathcal{B}_R}(A) = \oplus_{w\in\mathcal{B}_R}G^{\mathcal{B}_R}(A)_w$ with $G^{\mathcal{B}_R}(A)_w = F^{\mathcal{B}_R}_wA/F^{\mathcal{B}_R}_{\prec w}A$ the associated $\mathcal{B}_R$-graded algebra of A determined by $F^{\mathcal{B}_R}A$, where $F^{\mathcal{B}_R}_{\prec w}A = \cup_{u\prec w}F^{\mathcal{B}_R}_uA$. We have the following analogue of ([Li3], Theorem 1.1). Since the proof of this result is similar to that given in loc. cit., we omit it here.

3.1. Theorem With notation as above, there are graded R-algebra isomorphisms:

$$A^{\mathbb{N}}_{\rm LH} = R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle \cong G^{\mathbb{N}}(A),\quad A^{\mathcal{B}_R}_{\rm LH} = R\langle X\rangle/\langle \mathbf{LH}_{\mathcal{B}_R}(I)\rangle \cong G^{\mathcal{B}_R}(A).$$

$\square$

Since we are using an arbitrary commutative ring R (instead of a field) as the coefficient ring, the next lemma makes the *key bridge* for us to generalize the working principle of [LWZ] and [Li3] to quotient algebras of $R\langle X\rangle$ defined by monic Gröbner bases.

3.2. Lemma Let $R\langle X\rangle$ be equipped with the fixed weight $\mathbb{N}$-gradation as before, and I an ideal of $R\langle X\rangle$. Put $J = \langle \mathbf{LH}_{\mathbb{N}}(I)\rangle$. The following two statements hold.

(i) If h is a nonzero homogeneous element of $R\langle X\rangle$, then $h\in J$ if and only if $h\in\mathbf{LH}_{\mathbb{N}}(I)$. Hence $\mathbf{LH}_{\mathbb{N}}(J) = \mathbf{LH}_{\mathbb{N}}(I)$.

(ii) Let $\prec_{gr}$ be an $\mathbb{N}$-graded monomial ordering on $\mathcal{B}_R$ with respect to the fixed weight $\mathbb{N}$-gradation of $R\langle X\rangle$. Then $\mathbf{LH}_{\mathcal{B}_R}(J) = \mathbf{LH}_{\mathcal{B}_R}(I)$ and $\mathbf{LM}(J) = \mathbf{LM}(I)$.

(iii) Let $\prec_{gr}$ be an $\mathbb{N}$-graded monomial ordering on $\mathcal{B}_R$ with respect to the fixed weight $\mathbb{N}$-gradation of $R\langle X\rangle$. If $\mathcal{G}$ is a monic Gröbner basis of I, then

$$\langle \mathbf{LH}_{\mathcal{B}_R}(J)\rangle = \langle \mathbf{LH}_{\mathcal{B}_R}(I)\rangle = \langle \mathbf{LM}(\mathcal{G})\rangle = \langle \mathbf{LM}(I)\rangle = \langle \mathbf{LM}(J)\rangle.$$

Proof (i) Let h be a nonzero homogeneous element in $R\langle X\rangle$. If $h\in J$, then

$$h=\sum_{i,j}H_{ij}\mathbf{LH}_{\mathbb{N}}(f_i)T_{ij},\ \text{where } H_{ij},\ T_{ij} \text{ are homogeneous elements and } f_i\in I.$$

If we write $f_i=\mathbf{LH}_{\mathbb{N}}(f_i)+f_i'$, where $\deg(f_i')<\deg(f_i)$, then $f=\sum_{i,j}H_{ij}f_iT_{ij}\in I$ and

$$f=\sum_{ij}H_{ij}\mathbf{LH}_{\mathbb{N}}(f_i)T_{ij}+\sum_{i,j}H_{ij}f_i'T_{ij}=h+\sum_{i,j}H_{ij}f_i'T_{ij}.$$

It follows immediately that $h=\mathbf{LH}_{\mathbb{N}}(f)\in\mathbf{LH}(I)$. This shows that $\mathbf{LH}_{\mathbb{N}}(J)\subseteq\mathbf{LH}_{\mathbb{N}}(I)$ and hence the equality holds.

(ii) Note that $\prec_{gr}$ is an $\mathbb{N}$-graded monomial ordering on $\mathcal{B}_R$, every element of $\mathcal{B}_R$ is an $\mathbb{N}$-homogeneous element, and thus for $f\in R\langle X\rangle$ we have

$$(*)\qquad \mathbf{LH}_{\mathcal{B}_R}(f)=\mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(f)) \text{ and } \mathbf{LM}(f)=\mathbf{LM}(\mathbf{LH}_{\mathbb{N}}(f))$$

It follows from (i) and the above formula $(*)$ that

$$\begin{aligned}\mathbf{LH}_{\mathcal{B}_R}(J)=\mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(J))=\mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(I))=\mathbf{LH}_{\mathcal{B}_R}(I),\\ \mathbf{LM}(J)=\mathbf{LM}(\mathbf{LH}_{\mathcal{B}_R}(J))=\mathbf{LM}(\mathbf{LH}_{\mathcal{B}_R}(I))=\mathbf{LM}(I).\end{aligned}$$

(iii) Let $f\in R\langle X\rangle$ be a monic element with respect to the fixed monomial ordering $\prec_{gr}$, say $f=w+\sum\lambda_iw_i$ with $w,w_i\in\mathcal{B}_R$, $\lambda_i\in R$ and $\mathbf{LM}(f)=w$. Then it is clear that

$$(**)\qquad \mathbf{LM}(f)=w=\mathbf{LH}_{\mathcal{B}_R}(f).$$

So, if $\mathcal{G}$ is a monic Gröbner basis of I with respect to $\prec_{gr}$, then the above formula $(**)$ implies $\mathbf{LM}(\mathcal{G})=\mathbf{LH}_{\mathcal{B}_R}(\mathcal{G})\subset\mathbf{LH}_{\mathcal{B}_R}(I)$. Hence, by (ii) and Proposition 1.2 we obtain the desired equalities:

$$\langle\mathbf{LH}_{\mathcal{B}_R}(J)\rangle=\langle\mathbf{LH}_{\mathcal{B}_R}(I)\rangle=\langle\mathbf{LM}(\mathcal{G})\rangle=\langle\mathbf{LM}(I)\rangle=\langle\mathbf{LM}(J)\rangle.$$

□

Next, we show that an analogue of ([LWZ], Theorem 2.3.2 (i) $\Leftrightarrow$ (iii)) holds true for monic Gröbner bases in $R\langle X\rangle$.

3.3. Theorem Let I be an ideal of $R\langle X\rangle$. With notation as above, if $\prec_{gr}$ is an $\mathbb{N}$-graded monomial ordering on $\mathcal{B}_R$ with respect to a fixed weight $\mathbb{N}$-gradation of $R\langle X\rangle$, the following two statements are equivalent for a subset $\mathcal{G}\subset I$:

(i) $\mathcal{G}$ is a monic Gröbner basis of I ;

(ii) $\mathbf{LH}_{\mathbb{N}}(\mathcal{G})=\{\mathbf{LH}_{\mathbb{N}}(g)\mid g\in\mathcal{G}\}$ is a monic Gröbner basis for the $\mathbb{N}$-graded ideal $\langle\mathbf{LH}_{\mathbb{N}}(I)\rangle$.

Proof Since we are using the $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on $\mathcal{B}_R$, by Lemm 3.2 or its proof, a subset $\mathcal{G}$ of $R\langle X\rangle$ is monic if and only if $\mathbf{LH}_{\mathbb{N}}(\mathcal{G})$ is monic, and we have

$$\langle\mathbf{LM}(I)\rangle=\langle\mathbf{LM}(\mathcal{G})\rangle \text{ if and only if } \langle\mathbf{LM}(\langle\mathbf{LH}_{\mathbb{N}}(I)\rangle)\rangle=\langle\mathbf{LM}(\mathbf{LH}_{\mathbb{N}}(\mathcal{G}))\rangle.$$

It follows from Proposition 1.2 that $\mathcal{G}$ is a monic Gröbner basis for the ideal I if and only if $\mathbf{LH}_{\mathbb{N}}(\mathcal{G})$ is a monic Gröbner basis for the $\mathbb{N}$-graded ideal $\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle$, proving the equivalence of (i) and (ii). $\square$

Remark Also we point out that an analogue of ([LWZ], Theorem 2.3.2 (i) $\Leftrightarrow$ (ii)) works well for monic Gröbner bases when the homogenization of I in $R\langle X\rangle[t]$ is considered.

Combining the previous 3.1 – 3.3, we get immediately the result presenting the associated $\mathbb{N}$-graded algebra, respectively the associated $\mathcal{B}_R$-graded algebra via a monic Gröbner basis.

3.4. Theorem Let $R\langle X\rangle$ be equipped with a fixed weight $\mathbb{N}$-gradation as before, and I an ideal of $R\langle X\rangle$. If $\mathcal{G}$ is a monic Gröbner basis of I with respect to an $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on $\mathcal{B}_R$, then we have the graded algebra isomorphisms

$$\begin{aligned}
A_{\mathrm{LH}}^{\mathbb{N}} &= R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle = R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle \cong G^{\mathbb{N}}(A),\\
A_{\mathrm{LH}}^{\mathcal{B}_R} &= R\langle X\rangle/\langle \mathbf{LH}_{\mathcal{B}_R}(I)\rangle = R\langle X\rangle/\langle \mathbf{LM}(\mathcal{G})\rangle \cong G^{\mathcal{B}_R}(A),\\
(A_{\mathrm{LH}}^{\mathbb{N}})_{\mathrm{LH}}^{\mathcal{B}_R} &= R\langle X\rangle/\langle \mathbf{LH}_{\mathcal{B}_R}\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle\rangle = R\langle X\rangle/\langle \mathbf{LM}(\mathcal{G})\rangle \cong G^{\mathcal{B}_R}(A_{\mathrm{LH}}^{\mathbb{N}}).
\end{aligned}$$

$\square$

As in [Li3] we call the graded algebra isomorphisms presented in the last theorem the $\mathbb{N}$-*PBW isomorphism* (for the first one) and $\mathcal{B}_R$-*PBW isomorphism* (for the last two), determined by the given monic Gröbner basis $\mathcal{G}$ respectively.

Focusing on the first isomorphism of Theorem 3.4, typical examples can be given by using the Gröbner defining relations of Weyl algebras and enveloping algebras of Lie algebras, or more generally, the Gröbner defining relations of q-enveloping algebras determined in Example 3 of the last section, over a commutative ring. Here we specify several other examples. In all examples given below, R is an arbitrary commutative ring.

Example 1. Let $X = \{X_i\}_{i\in J}$ and $\mathsf{C} = R\langle X\rangle/\langle \mathcal{G}\rangle$ the Clifford algebra over R, where $\mathcal{G}$ consists of

$$\begin{aligned}
g_i &= X_i^2 - q_i, && i\in J,\ q_i\in R,\\
g_{k\ell} &= X_kX_\ell + X_\ell X_k - q_{k\ell}, && k,\ell\in J,\ k>\ell,\ q_{k\ell}\in R.
\end{aligned}$$

Note that if all the $q_i = 0$, $q_{k,\ell} = 0$, we get the defining relations of an R-exterior algebra. It is well known that if $R = K$ is a field, then, under the $\mathbb{N}$-graded lexicographic ordering $\prec_{gr}$ such that $\deg X_i = 1$, $i\in J$, and

$$X_\ell \prec_{gr} X_k,\quad \ell,k\in J,\ \ell<k,$$

$\mathcal{G}$ forms a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$ (e.g., see CH.II of [Li2]). It follows from Proposition 1.6 that $\mathcal{G}$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$. By Theorem 3.4, with respect

to the natural $\mathbb{N}$-filtration $F^{\mathbb{N}}\mathsf{C}$ of C, the associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(\mathsf{C}) \cong R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle$ of C is nothing but an exterior algebra E over R.

Example 2. Let $A = R\langle X_1, X_2\rangle/\langle \mathcal{G}\rangle$ be a down-up R-algebra in the sense of [Ben], where $\mathcal{G}$ consists of

$$\begin{aligned} g_1 &= X_1^2X_2 - \alpha X_1X_2X_1 - \beta X_2X_1^2 - \gamma X_1, \\ g_2 &= X_1X_2^2 - \alpha X_2X_1X_2 - \beta X_2^2X_1 - \gamma X_2, \end{aligned} \qquad \alpha, \beta \in R.$$

It is well known that if $R = K$ is a field, then, under the $\mathbb{N}$-graded lexicographic ordering $\prec_{gr}$ such that $\deg X_1 = \deg X_2 = 1$ and $X_2 \prec_{gr} X_1$, $\mathcal{G}$ forms a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$ (e.g., see CH.II of [Li2]). It follows from Proposition 1.6 that $\mathcal{G}$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$. By Theorem 3.4, with respect to the natural $\mathbb{N}$-filtration $F^{\mathbb{N}}A$ of A, the associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle$ of A is a down-up algebra over R with the set of defining relations $\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{\mathbf{LH}_{\mathbb{N}}(g_1) = X_1^2X_2 - \alpha X_1X_2X_1 - \beta X_2X_1^2,\ \mathbf{LH}_{\mathbb{N}}(g_2) = X_1X_2^2 - \alpha X_2X_1X_2 - \beta X_2^2X_1\}$; in particular, one sees that if $\alpha = 2$ and $\beta = -1$, then $G^{\mathbb{N}}(A)$ is nothing but the universal enveloping algebra of the $(-)$-part (or $(+)$-part) of the Kac-Moody R-Lie algebra associated to the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Example 3. Let $A = R\langle X_1, X_2\rangle/\langle g_{21}\rangle$ be the R-algebra as given in (Section 2, Example 5). Then by Theorem 3.4, with respect to both the natural $\mathbb{N}$-filtration and the weight $\mathbb{N}$-filtration induced by the weight $\mathbb{N}$-grading filtration of $R\langle X_1, X_2\rangle$, A has the associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2\rangle/\langle X_2X_1 - qX_1X_2\rangle$, which, in the case that q is invertible, is the coordinate ring of the quantum plane over R.

Example 4. Let $A = R\langle X_1, X_2, X_3\rangle/\langle \mathcal{G}\rangle$ be the R-algebra as given in (Section 2, Example 6). In the case that $f(X_1)$ has degree ≤ 2, i.e., $f(X_1)$ is of the form

$$f(X_1) = aX_1^2 + bX_1 + c \text{ with } a, b, c \in R,$$

then by Theorem 3.4, with respect to the natural $\mathbb{N}$-filtration $F^{\mathbb{N}}A$, A has the associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2, X_3\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle$ with

$$\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{X_3X_1 - \lambda X_1X_3,\ X_1X_2 - \lambda X_2X_1,\ X_3X_2 - \omega X_2X_3 + aX_1^2\};$$

while in the case that $f(X_1)$ has degree $n \ge 3$, if the weight $(1, n, n)$ is used, then by Theorem 3.4, with respect to the weight $\mathbb{N}$-filtration $F^{\mathbb{N}}A$ induced by the weight $\mathbb{N}$-grading filtration of $R\langle X_1, X_2\rangle$, A has the associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2, X_3\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle$ with

$$\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{X_3X_1 - \lambda X_1X_3,\ X_1X_2 - \lambda X_2X_1,\ X_3X_2 - \omega X_2X_3\}$$

By referring to the well-known filtered-graded comparison principle for algebras with an $\mathbb{N}$-filtration ([MR], [Li1], [LVO1], [Li3]), we now summarize, without proof, several applications of

Theorem 3.3 and Theorem 3.4. Let R be an arbitrary commutative ring. For the convenience, in what follows we let the free R-algebra $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ be equipped with a fixed weight $\mathbb{N}$-gradation, $I = \langle \mathcal{G}\rangle$ an ideal of $R\langle X\rangle$ generated by a monic Gröbner basis $\mathcal{G}$ with respect to an $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, and $A = R\langle X\rangle/I$. Then the following diagram may indicate how all results to be given will work:

$$\begin{tikzcd}
 & A = R\langle X\rangle/\langle \mathcal{G}\rangle & & \\
G^{\mathcal{B}_R}(A) \arrow[ur, "\text{lifting}"] & & G^{\mathbb{N}}(A) \arrow[ul, "\text{lifitng}"'] & \dfrac{R\langle X\rangle}{\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle} = A^{\mathbb{N}}_{\mathrm{LH}} \arrow[l, "\cong", "\mathbb{N}-\mathrm{PBW}"'] \\
A^{\mathcal{B}_R}_{\mathrm{LH}} = \dfrac{R\langle X\rangle}{\langle \mathbf{LM}(\mathcal{G})\rangle} \arrow[u, "\cong"', "\mathcal{B}_R-\mathrm{PBW}"] \arrow[rrr, "\cong", "\mathcal{B}_R-\mathrm{PBW}"'] & & & G^{\mathcal{B}_R}(A^{\mathbb{N}}_{\mathrm{LH}}) \arrow[u, "\text{lifting}"']
\end{tikzcd}$$

3.5. Theorem Under the respective canonical algebra epimorphism, the set $N(\mathcal{G})$ of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$), projects to a free R-basis for the algebras $A = R\langle X\rangle/I$, $A^{\mathbb{N}}_{\mathrm{LH}} = R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(I)\rangle$, and $A^{\mathcal{B}_R}_{\mathrm{LH}} = R\langle X\rangle/\langle \mathbf{LM}(I)\rangle$ respectively, and thereby to a free R-basis for $G^{\mathbb{N}}(A)$, $G^{\mathcal{B}_R}(A)$, and $G^{\mathcal{B}_R}(A^{\mathbb{N}}_{\mathrm{LH}}(A))$, respectively. $\square$

3.6. Theorem Bearing $A^{\mathcal{B}_R}_{\mathrm{LH}} = R\langle X\rangle/\langle \mathbf{LM}(\mathcal{G})\rangle$ in mind, the following statements hold.

(i) If $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is a (semi-)prime ring, then $A^{\mathbb{N}}_{\mathrm{LH}}$ is a (semi-)prime ring (hence $G^{\mathbb{N}}(A)$ is a (semi-)prime ring), and A is a (semi-)prime ring.

(ii) If $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is $\mathcal{B}_R$-graded left Noetherian, that is, every $\mathcal{B}_R$-graded left ideal of $G(A)$ is finitely generated, then $A^{\mathbb{N}}_{\mathrm{LH}}$ is left Noetherian (hence $G^{\mathbb{N}}(A)$ is left Noetherian), and A is left Noetherian.

(iii) If $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is $\mathcal{B}_R$-graded left Artinian, that is, $A^{\mathcal{B}_R}_{\mathrm{LH}}$ satisfies the descending chain condition for $\mathcal{B}_R$-graded left ideals, then $A^{\mathbb{N}}_{\mathrm{LH}}$ is left Artinian (hence $G^{\mathbb{N}}(A)$ is left Artinian), and A is left Artinian.

(iv) If $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is a $\mathcal{B}_R$-graded simple R-algebra, that is, $A^{\mathcal{B}_R}_{\mathrm{LH}}$ does not have nontrivial $\mathcal{B}_R$-graded ideal, then $A^{\mathbb{N}}_{\mathrm{LH}}$ is a simple R-algebra (hence $G^{\mathbb{N}}(A)$ is a simple R-algebra), and A is a simple R-algebra.

(v) If the Krull dimension (K.dim in the sense of Gabriel and Rentschler, e.g. see [MR] for the definition) of $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is well-defined, then the Krull dimension of $A^{\mathbb{N}}_{\mathrm{LH}}$ (hence of $G^{\mathbb{N}}(A)$) and A is defined and $\mathrm{K.dim}A \le \mathrm{K.dim}\ G^{\mathbb{N}}(A) = \mathrm{K.dim}A^{\mathbb{N}}_{\mathrm{LH}} \le \mathrm{K.dim}A^{\mathcal{B}_R}_{\mathrm{LH}}$.

(vi) If $A^{\mathcal{B}_R}_{\mathrm{LH}}$ is semisimple (simple) Artinian, then $A^{\mathbb{N}}_{\mathrm{LH}}$ is semisimple (simple) Artinian (hence $G^{\mathbb{N}}(A)$ is semisimple (simple) Artinian), and A is semisimple (simple) Artinian.

(vii) Let gl.dim abbreviate the phrase "global homological dimension". We have $\mathrm{gl.dim}A \le \mathrm{gl.dim}G^{\mathbb{N}}(A) = \mathrm{gl.dim}A^{\mathbb{N}}_{\mathrm{LH}} \le \mathrm{gl.dim}A^{\mathcal{B}_R}_{\mathrm{LH}}$.

(viii) If $A_{\rm LH}^{\mathcal{B}_R}$ is left hereditary, then $A_{\rm LH}^{\mathbb{N}}$ is left hereditary (hence $G^{\mathbb{N}}(A)$ is left hereditary), and A is left hereditary.
(ix) Let gl.wdim abbreviate the phrase "global week homological dimension". We have gl.wdim$A\le$ gl.wdim$G^{\mathbb{N}}(A)=$ gl.wdim$A_{\rm LH}^{\mathbb{N}}\le$ gl.wdim$A_{\rm LH}^{\mathcal{B}_R}$.
(x) If $A_{\rm LH}^{\mathcal{B}_R}$ is a Von Neuman regular ring, then $A_{\rm LH}^{\mathbb{N}}$ is Von Neuman regular ring (hence $G^{\mathbb{N}}(A)$ is a Von Neuman regular ring), and A is a Von Neuman regular ring. □

3.7. Theorem Bearing $A_{\rm LH}^{\mathbb{N}}=R\langle X\rangle/\langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G})\rangle$ in mind, if the role of $A_{\rm LH}^{\mathcal{B}_R}$ is replaced by $A_{\rm LH}^{\mathbb{N}}$, then the analogues of Theorem 4.6 (i) – (x) hold true. Moreover, we have:
(i) If $A_{\rm LH}^{\mathbb{N}}$ is a domain, then A is a domain.
(ii) If $A_{\rm LH}^{\mathbb{N}}$ is a Noetherian domain and maximal order in its quotient ring (see e.g. [MR] for the definition), then A is a Noetherian domain and maximal order in its quotient ring.
(iii) If $A_{\rm LH}^{\mathbb{N}}$ is an Auslander regular ring (see e.g. [Li1], [LVO] for the definition), then A is an Auslander regular ring. □

References

[An] D. J. Anick, On the homology of associative algebras, *Trans. Amer. Math. Soc.*, 2(296)(1986), 641–659.

[Ben] G. Benkart, Down-up algebras and Witten's deformations of the universal enveloping algebra of sl_2, *Contemp. Math.*, 224(1999), 29–45.

[Ber] R. Berger, The quantum Poincaré-Birkhoff-Witt theorem, *Comm. Math. Physics*, 143(1992), 215–234.

[Ber1] G. Bergman, The diamond lemma for ring theory, *Adv. Math.*, 29(1978), 178–218.

[BM] L. A. Bokut and P. Malcolmson, Grobner-Shirshov basis for quantum enveloping algebras, *Israel Journal of Mathematics*, 96(1996), 97–113.

[CE] B.L. Cox and T.J. Enright, Representations of quantum groups defined over commutative rings, *Comm. Alg*, 23(6)(1995), 2215 - 2254.

[CSS] Y. Chen, H. Shao and K. P. Shum, Rosso-Yamane Theorem on PBW basis of $U_q(A_N)$, *CUBO, A Mathematical Journal*, 10(3)(2008), 171–194. arXiv:0804.0954v1 [math.RA]

[CU] S. Cojocaru and V. Ufnarofski, BERGMAN under MS-DOS and Anick's resolution, *Discrete Mathematics and Theoretical Computer Science*, 1(1997), 139–147.

[EPS] D. Eisenbud, I. Peeva and B. Sturmfels, Non-commutative Gröbner bases for commutative algebras, *Proc. Amer. Math. Soc.*, 126(1998), 687–691.

[GI-L] T. Gateva-Ivanova and V. Latyshev, On recognizable properties of associative algebras, *J. Symbolic Computation*, 6(1988), 371–388.

[Gol] E. S. Golod, On noncommutative Gröbner bases over rings, *Journal of Mathematical Science*, 140(2)(2007), 239–242.

[Gr] E. L. Green, Noncommutative Grobner bases and projective resolutions, in: *Proceedings*

of the Euroconference Computational Methods for Representations of Groups and Algebras, Essen, 1997, (Michler, Schneider, eds), Progress in Mathematics, Vol. 173, Basel, Birkhauser Verlag, 1999, 29–60.

[Jac] N. Jacobson, *Lie Algebras*, New York, Wiley, 1962.

[LB] L. Le Bruyn, Conformal sl_2 enveloping algebras, *Comm. Alg.*, 23(1995), 1325–1362.

[Li1] H. Li, *Noncommutative Zariskian Filtered Rings*, PhD Thesis, Antwerp University, 1989.

[Li2] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*, LNM, 1795, Springer-Verlag, 2002.

[Li3] H. Li, Γ-leading homogeneous algebras and Gröbner bases, in: *Advanced Lectures in Mathematics*, Vol.8, International Press, Boston, 2009, 155–200.

[Li4] H. Li, Looking for Grobner basis theory for (almost) skew 2-nomial algebras, *Journal of Symbolic Computation*, 45(2010), in press, online 10 may 2010, doi:10.1016/j.jsc.2010.05.002. (also see arXiv:math.RA/0808.1477)

[LVO1] H. Li and F. Van Oystaeyen, *Zariskian Filtrations*, K-Monograph in Math., Vol.2, Kluwer Academic Publishers, 1996.

[LVO2] H. Li and F. Van Oystaeyen, Reductions and global dimension of quantized algebras over a regular commutative domain, *Comm. Alg.*, 26(4)(1998), 1117–1124.

[LWZ] H. Li, Y. Wu and J. Zhang, Two applications of noncommutative Gröbner bases, *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.*, XLV(1999), 1–24.

[Man] Yu.I. Manin, *Quantum Groups and Noncommutative Geometry*, Les Publ. du Centre de Récherches Math., Universite de Montreal, 1988.

[MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.

[Mor] T. Mora, An introduction to commutative and noncommutative Gröbner bases, *Theoretic Computer Science*, 134(1994), 131–173.

[Rin] C. M. Ringel, PBW-bases of quantum groups, *J. Reine Angew.* Math. 470 (1996), 51–88.

[Ros] M. Rosso, An analogue of the Poincare-Birkhoff-Witt theorem and the universal R-matrix of $U_q(sl(N+1))$, *Comm. Math. Phys.*, 124(2)(1989), 307–318.

[Uf1] V. Ufnarovski, A growth criterion for graphs and algebras defined by words, *Mat. Zametki*, 31(1982), 465–472 (in Russian); English translation: *Math. Notes*, 37(1982), 238–241.

[Uf2] V. Ufnarovski, On the use of graphs for computing a basis, growth and Hilbert series of associative algebras, (in Russian 1989), *Math. USSR Sbornik*, 11(180)(1989), 417-428.

[Yam] I. Yamane, A Poincare-Birkhoff-Witt theorem for quantized universal enveloping algebras of type A_N, Publ., *RIMS. Kyoto Univ.*, 25(3)(1989), 503–520.